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A NOTE ON A REGULARIZED GAP FUNCTION OF QVI IN BANACH SPACES

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Abstract. Recently, Taji [7] and Harms et al. [4] studied the regularized gap function of QVI analogous to that of VI by Fukushima [2]. Discussions are made in a finite dimensional Euclidean space. In this note, an infinite dimensional generalization is considered in the framework of a reflexive Banach space. To do so, we introduce an extended quasi-variational inequality problem (in short, EQVI) and a generalized regularized gap function of EQVI. Then we investigate some basic properties of it. Our results may be regarded as an infinite dimensional extension of corresponding results due to Taji [7].

1. Introduction

Given a function $F : \mathbb{R}^n \to \mathbb{R}^n$ and a multifunction $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with closed convex values Sx for all $x \in \mathbb{R}^n$, the quasi-variational inequality problem (in short, QVI) is to find a vector $\bar{x} \in S\bar{x}$ such that

$$\langle F\bar{x}, y - \bar{x}\rangle \geq 0 \quad \text{for all } y \in S\bar{x}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denote the inner product in $\mathbb{R}^n$. If C is a convex closed subset of $\mathbb{R}^n$ and for all $x \in \mathbb{R}^n$, $Sx = C$, then QVI reduces to the standard variational inequality problem(VI). It is well-known that VI can be casted as the following equivalent problem:

$$\text{minimize } \phi(x) \quad \text{subject to } x \in C$$

where $\phi(x) = -\min_{y \in C} \langle Fx, y - x\rangle + \frac{1}{2}\|y - x\|^2$ is the regularized gap function by Fukushima [2]. One of the most important features, besides some excellent global properties, of the regularized gap function ϕ is

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that it is continuously differentiable if F is so. At this point, a natural question arises: Is this approach still available for QVI? The answer is affirmative since Giannessi [3] proposed a gap function for QVI as follows:

$$g(x) := -\inf_{y \in Sx} \langle Fx, y - x \rangle.$$

But his original gap function is nondifferentiable, possibly extended-valued. So it is not suitable for dealing with optimization problems. To resolve this drawback, an extension of the regularized gap function by Fukushima [2] to QVIs is provided by Taji [7] as follows.

$$g_\alpha(x) := -\min_{y \in Sx} \langle Fx, y - x \rangle + \frac{\alpha}{2} \|y - x\|^2$$

where $\alpha > 0$ is a parameter. Taji [7] verified characteristic properties of the regularized gap function g_α. However, being different from the case of VI, g_α has no nice differentiablity as ϕ does. So, recently, Harms et al. [4] exploited Taji's approach to take a closer look at the nondifferentiable points so that they could specialize the results to generalized Nash equilibrium problems. Of course, in [7, 4] only finite dimensional cases are treated.

Motivated by this observation, in this note, we consider the following question from a theoretical view point: What happens in an infinite dimensional case? We introduce an extended quasi-variational inequality problem (in short, EQVI) and a generalized regularized gap function of EQVI. Then we investigate some basic properties of it in a reflexive Banach space. Our results may be regarded as an infinite dimensional extension of corresponding results due to Taji [7].

2. Preliminaries

Consider the following extended quasi-variational inequality EQVI.

EQVI : Let E be a real Banach space and E^* be its dual space. Given a function $F : E \to E^*$, a multifunction $S : E \rightrightarrows E$ with closed convex values Sx for all $x \in E$ (for the simplicity of argument, Sx is assumed to be nonempty), and a convex and continuous function $f : E \to \mathbb{R}$, find $\bar{x} \in S\bar{x}$ such that

$$\langle F\bar{x}, x - \bar{x} \rangle \geq f(\bar{x}) - f(x) \quad \text{for all } x \in S\bar{x}, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denote the dual paring on $E \times E^*$. If f is the zero function, EQVI (2.1) becomes QVI. The notion EQVI is adopted from Kum and Lee [6] (originally, Chen et al. [1]).

Throughout this paper, we make the following assumptions.

ASSUMPTION 2.1. Let $\Omega : E \times E \to \mathbb{R}$ be nonnegative and for each $x \in E$, $\Omega(x, \cdot)$ be continuously Gâteaux differentiable (not necessarily convex) on E. Assume that $\Omega(x, y) = 0$ if and only if $x = y$ and $\nabla_y \Omega(x, x) = 0$ for any $x \in E$.

Our concern is to show that the function $\phi : E \to \mathbb{R}$ is a continuous gap function for EQVI (2.1) under suitable conditions;

$$\phi(x) := -\inf_{y \in Sx} \{\langle F(x), y - x\rangle + f(y) - f(x) + \Omega(x, y)\}. \tag{2.2}$$

Before going to the main result, it is necessary to recall continuity notions of multifunctions in Harms et al. [4, Definition 3.1].

DEFINITION 2.2. Let $X \subseteq E_1$, $Y \subseteq E_2$, and $T : X \rightrightarrows Y$ be a multifunction where E_1, E_2 are Banach spaces. Then T is called

(i) lower semicontinuous at $\bar{x} \in X$ if for all sequences $\{x^k\} \subseteq X$ with $x^k \to \bar{x}$ and all $\bar{y} \in T\bar{x}$, there exists $k_0 \in \mathbb{N}$ and a sequence $\{y^k\} \subseteq Y$ with $y^k \to \bar{y}$ and $y^k \in Tx^k$ for all $k \geq k_0$;

(ii) closed at $\bar{x} \in X$ if for all sequences $\{x^k\} \subseteq X$ with $x^k \to \bar{x}$ and all sequences $\{y^k\} \to \bar{y}$ with $y^k \in Tx^k$ for all $k \in \mathbb{N}$ sufficiently large, we have $\bar{y} \in T\bar{x}$;

(iii) continuous at at $\bar{x} \in X$ if it is lower semicontinuous and closed at $\bar{x} \in X$;

(iv) lower semicontinuous, closed or continuous on X if it is lower semicontinuous, closed or continuous at every $x \in X$, respectively.

3. Main result

THEOREM 3.1. *The function ϕ in (2.2) is a gap function of EQVI, that is, it satisfies*

(i) $\phi(x) \geq 0$ *for all* $x \in Sx$;

(ii) $\bar{x}$ *is a solution of EQVI if and only if* $\bar{x} \in S\bar{x}$ *and* $\phi(\bar{x}) = 0$.

Proof. (i) Taking $y = x$ in (2.2) yields the result.

(ii) ($\Rightarrow$) If $\bar{x}$ solves EQVI, then we have

$$\langle F\bar{x}, x - \bar{x}\rangle \geq f(\bar{x}) - f(x) \quad \text{for all } x \in S\bar{x}.$$

As Ω is nonnegative on $E \times E$,

$$\langle F\bar{x}, x - \bar{x} \rangle \geq f(\bar{x}) - f(x) - \Omega(\bar{x}, x) \quad \text{for all } x \in S\bar{x}.$$

Hence

$$\langle F\bar{x}, x - \bar{x} \rangle + f(x) - f(\bar{x}) + \Omega(\bar{x}, x) \geq 0 \quad \text{for all } x \in S\bar{x}.$$

This implies that $\phi(\bar{x}) \leq 0$. Obviously, $\bar{x} \in S\bar{x}$ and $\phi(\bar{x}) \geq 0$ by (i). Thus $\phi(\bar{x}) = 0$.

($\Leftarrow$) Assume that $\bar{x} \in S\bar{x}$ and $\phi(\bar{x}) = 0$. Define a function

$$\tilde{\phi}(x) = -\inf_{y \in S\bar{x}} \{\langle Fx, y - x \rangle + f(y) - f(x) + \Omega(x, y)\} \quad \text{for all } x \in E.$$

In fact, this function $\tilde{\phi}(x)$ is nothing but the gap function of EVI in Kum and Lee [6] for the fixed closed convex subset $S\bar{x}$ of E. Also, it is obvious that $\tilde{\phi}(x) \geq 0$ for all $x \in S\bar{x}$. Moreover, $\tilde{\phi}(\bar{x}) = \phi(\bar{x}) = 0$ and $\bar{x} \in S\bar{x}$. This means that $\bar{x}$ is a solution of EVI by virtue of Kum and Lee [6, Theorem 3.1]. That is, $\langle F\bar{x}, x - \bar{x} \rangle \geq f(\bar{x}) - f(x)$ for all $x \in S\bar{x}$, which amounts to saying that $\bar{x}$ is a solution of EQVI, as desired. □

For more discussions of ϕ, from now on, it is further assumed that E is a reflexive Banach space and $\Omega(x, \cdot)$ is strongly convex in the second variable for each $x \in E$. Recall that a function $g : E \to \mathbf{R}$ is said to be *strongly convex* with modulus a ($a > 0$) if for all $x,\ y \in E$, and $\alpha \in [0, 1]$, we have

$$g(\alpha x + (1 - \alpha)y) \leq \alpha g(x) + (1 - \alpha)g(y) - \frac{1}{2} a\alpha(1 - \alpha)\|x - y\|^2.$$

Thus for each fixed x, the function

$$\psi(y) = \langle Fx, y - x \rangle + f(y) - f(x) + \Omega(x, y)$$

is strongly convex with respect to y, so it is coercive in the sense that $\lim_{\|y\| \to +\infty} \frac{\psi(y)}{\|y\|} \geq +\infty$. (For a proof, see Kum and Lee [6, Proposition 4.2].) As the closed unit ball B in E is weakly compact, $\psi(y)$ has a unique minimizer $z(x) \in Sx$ over the closed convex subset Sx so that

$$\begin{aligned} \phi(x) &= -\min_{y \in Sx} \psi(y) = -\psi(z(x)) \\ &= -\langle Fx, z(x) - x \rangle - f(z(x)) + f(x) - \Omega(x, z(x)). \end{aligned} \tag{3.1}$$

Then a solution $\bar{x}$ of EQVI has the following fixed point characterization analogous to the case of VI [2]:

THEOREM 3.2. *$\bar{x}$ is a solution of EQVI if and only if $\bar{x} = z(\bar{x})$.*

Proof. (⇒) Let $\bar{x}$ be a solution of EQVI. Since $z(\bar{x})$ is a minimizer of the convex function $\psi(y)$ over the closed convex set $S\bar{x}$, we have

$$0 \in F\bar{x} + \partial f(z(\bar{x})) + \nabla_y \Omega(\bar{x}, z(\bar{x})) + N_{S\bar{x}}(z(\bar{x})) \tag{3.2}$$

where $\partial f(z(\bar{x}))$ is the subdifferential of f at $z(\bar{x})$ and $N_{S\bar{x}}(z(\bar{x})) = \{x^* \in E^* \mid \langle x^*,\ x - z(\bar{x})\rangle \leq 0 \text{ for all } x \in S\bar{x}\}$ is the normal cone of $S\bar{x}$ at $z(\bar{x})$. Hence there exists $x^* \in \partial f(z(\bar{x}))$ such that

$$-F\bar{x} - x^* - \nabla_y \Omega(\bar{x}, z(\bar{x})) \in N_{S\bar{x}}(z(\bar{x})).$$

That is,

$$\langle F\bar{x} + x^* + \nabla_y \Omega(\bar{x}, z(\bar{x})), x - z(\bar{x})\rangle \geq 0 \text{ for all } x \in S\bar{x}.$$

Taking $x = \bar{x} \in S\bar{x}$, we obtain

$$\begin{aligned}\langle x^* + \nabla_y \Omega(\bar{x}, z(\bar{x})), \bar{x} - z(\bar{x})\rangle \geq \langle F\bar{x}, z(\bar{x}) - \bar{x}\rangle &\geq f(\bar{x}) - f(z(\bar{x})) \\ &\geq \langle x^*, \bar{x} - z(\bar{x})\rangle\end{aligned}$$

because $\bar{x}$ is a solution of EQVI and $x^* \in \partial f(z(\bar{x}))$. Thus we get

$$\langle \nabla_y \Omega(\bar{x}, z(\bar{x})), \bar{x} - z(\bar{x})\rangle \geq 0. \tag{3.3}$$

From the strong convexity of $\Omega(\bar{x}, \cdot)$ with (3.3) it follows that

$$0 \leq \langle \nabla_y \Omega(\bar{x}, z(\bar{x})), \bar{x} - z(\bar{x})\rangle + \mu\|\bar{x} - z(\bar{x})\|^2 \leq \Omega(\bar{x}, \bar{x}) - \Omega(\bar{x}, z(\bar{x})) \leq 0$$

for some $\mu > 0$. Here the second inequality comes from Kum and Lee [6, (4.3) of Proposition 4.2]. This implies that $\Omega(\bar{x}, \bar{x}) - \Omega(\bar{x}, z(\bar{x})) = -\Omega(\bar{x}, z(\bar{x})) = 0$, hence $\bar{x} = z(\bar{x})$ by Assumption 2.1.

(⇐) Assume that $\bar{x} = z(\bar{x})$. By (3.2), for some $x^* \in \partial f(z(\bar{x}))$, we again obtain that

$$\langle F\bar{x} + x^* + \nabla_y \Omega(\bar{x}, z(\bar{x})), x - z(\bar{x})\rangle \geq 0 \text{ for all } x \in S\bar{x}.$$

Thus

$$\langle F\bar{x} + x^* + \nabla_y \Omega(\bar{x}, \bar{x}), x - \bar{x}\rangle \geq 0 \text{ for all } x \in S\bar{x}.$$

Hence

$$\langle F\bar{x} + x^*, x - \bar{x}\rangle \geq 0 \text{ for all } x \in S\bar{x}.$$

On the other hand,

$$\langle x^*, x - \bar{x}\rangle = \langle x^*, x - z(\bar{x})\rangle \leq f(x) - f(z(\bar{x})) = f(x) - f(\bar{x}).$$

Therefore

$$\langle F\bar{x}, x - \bar{x}\rangle \geq \langle x^*, \bar{x} - x\rangle \geq f(\bar{x}) - f(x) \text{ for all } x \in S\bar{x},$$

which means that $\bar{x}$ is a solution of EQVI. □

Now we deal with the continuity of the gap function ϕ of EQVI in terms of that of $z(x)$.

THEOREM 3.3. *Let the function $F : E \to E^*$ be continuous and let the multifunction $S : E \rightrightarrows E$ be continuous. Then the gap function $\phi : E \to \mathbb{R}$ has a closed graph in $E \times \mathbb{R}$.*

Proof. By Hogan [5, Theorem 8], $z : x \mapsto \{z(x)\}$ is closed on E, i.e., the function z has a closed graph in $E \times E$. It follows from (3.1) that $\phi(x) = -\langle Fx, z(x) - x\rangle - f(z(x)) + f(x) - \Omega(x, z(x))$ has the same property. $\square$

THEOREM 3.4. *Let F and S be as in Theorem 3.3. Assume that the image $S(E)$ is relatively compact. Then the gap function $\phi : E \to \mathbb{R}$ is continuous.*

Proof. By Hogan [5, Corollary 8.1], the function $z : E \to E$ is continuous, hence $\phi : E \to \mathbb{R}$ is continuous, too. $\square$

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